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# S-duality action on discrete T-duality invariants

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Abstract: In heterotic string theory compactified on $T^6$, the T-duality orbits of dyons of charge $(Q, P)$ are characterized by $O(6, 22; \mathbb{R})$ invariants $Q^2$, $P^2$ and $Q \cdot P$ together with a set of invariants of the discrete T-duality group $O(6, 22; \mathbb{Z})$. We study the action of S-duality group on the discrete T-duality invariants and study its consequence for the dyon degeneracy formula. In particular we find that for dyons with torsion $r$, the degeneracy formula, expressed as a function of $Q^2$, $P^2$ and $Q \cdot P$, is required to be manifestly invariant under only a subgroup of the S-duality group. This subgroup is isomorphic to $\Gamma^0(r)$. Our analysis also shows that for a given torsion $r$, all other discrete T-duality invariants are characterized by the elements of the coset $\mathrm{SL}(2, \mathbb{Z})/\Gamma^0(r)$.

Keywords: Black Holes in String Theory, String Duality.

JHEP04(2008)012

Dyons in heterotic string theory on $T^6$ are characterized by a pair of charge vectors $(Q, P)$ each taking value on the Narain lattice $\Lambda$ [1, 2]. Given two pairs of charge vectors, an interesting question is: under what condition can they be related via a T-duality transformation? This question was answered in [3] where a complete set of T-duality invariants classifying a pair of charge vectors $(Q, P)$ were constructed. These include the invariants of the continuous T-duality group $O(6, 22; \mathbb{R})$

$$Q^2, \quad P^2, \quad Q \cdot P\,, \tag{1}$$

together with a set of invariants of the discrete T-duality group $O(6, 22; \mathbb{Z})$. These are defined as follows. We shall assume that the dyon is primitive so that $(Q, P)$ cannot be written as an integer multiple of $(Q_0, P_0)$ with $Q_0, P_0 \in \Lambda$, but we shall not assume that $Q$ and $P$ themselves are primitive. Now consider the intersection of the two dimensional vector space spanned by $(Q, P)$ with the Narain lattice $\Lambda$. The result is a two dimensional lattice $\Lambda_0$. Let $(e_1, e_2)$ be a pair of basis elements whose integer linear combinations generate this lattice. We can always choose $(e_1, e_2)$ such that in this basis

$$\begin{aligned} &Q = r_1 e_1, \qquad P = r_2(u_1 e_1 + r_3 e_2), \qquad r_1, r_2, r_3, u_1 \in \mathbb{Z}^+, \\ &\gcd(r_1, r_2) = 1, \quad \gcd(u_1, r_3) = 1, \quad 1 \le u_1 \le r_3\,. \end{aligned} \tag{2}$$

It was found in [3] that besides $Q^2$, $P^2$ and $Q \cdot P$, the integers $r_1$, $r_2$, $r_3$ and $u_1$ are T-duality invariants. Furthermore it was found that this is the complete set of T-duality invariants. Thus a pair of charge vectors $(Q, P)$ can be transformed into another pair $(Q', P')$ via a T-duality transformation if and only if all the invariants agree for these two pairs.

Our first goal is to study some aspects of the action of the S-duality transformation

$$Q \to Q' = aQ + bP, \quad P \to P' = cQ + dP, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1\,, \tag{3}$$

on the invariants $r_1$, $r_2$, $r_3$ and $u_1$. Substituting (2) into (3), and expressing the resulting $(Q', P')$ as $(r_1' e_1', r_2'(u_1' e_1' + r_3' e_2'))$ for some primitive basis $(e_1', e_2')$ of $\Lambda_0$ we can determine $(r_1', r_2', r_3', u_1')$. Since the resuting expressions are somewhat complicated and not very illuminating we shall not describe them here. Instead we shall focus on some salient features of the transformation laws of $(r_1, r_2, r_3, u_1)$. We first note that the torsion $r(Q, P)$ associated with a pair of charges $(Q, P)$, defined as [4, 5]

$$r(Q, P) = Q_1 P_2 - Q_2 P_1\,, \tag{4}$$

with $Q_i, P_i$ being the components of $Q$ and $P$ along $e_i$, is invariant under the S-duality transformation (3). Furthermore, for the charge vectors $(Q, P)$ given in (2) we have

$$r(Q, P) = r_1 r_2 r_3\,. \tag{5}$$

We shall now show that one can always find an S-duality transformation that brings the T-duality invariants $(r_1, r_2, r_3, u_1)$ to $(r_1 r_2 r_3, 1, 1, 1)$ together with an appropriate transformation on $Q^2$, $P^2$ and $Q \cdot P$ induced by (3). For this we note that under the S-duality transformation (3), $(Q, P)$ given in (2) transforms to

$$\begin{aligned} Q' &= \{a r_1 + b r_2(u_1 + k r_3)\} e_1 + b r_2 r_3 (e_2 - k e_1), \\ P' &= \{c r_1 + d r_2(u_1 + k r_3)\} e_1 + d r_2 r_3 (e_2 - k e_1)\,, \end{aligned} \tag{6}$$



JHEP04(2008)012

where $k$ is an arbitrary integer. We shall choose

$$k = \prod_i p_i \,, \tag{7}$$

where $\{p_i\}$ represent the collection of primes which are factors of $r_1$ but not of $u_1$. Now we know from (2) that $\gcd(r_1, r_2) = 1$. On the other hand it follows from a result derived in appendix E of [6] that for the choice of $k$ given in (7) we have $\gcd(r_1, u_1 + kr_3) = 1$. Thus if we choose

$$b = r_1, \quad a = -r_2(u_1 + kr_3)\,, \tag{8}$$

we have $\gcd(a, b) = 1$ and hence we can always find $c$, $d$ satisfying $ad - bc = 1$. For this particular choice of $\mathrm{SL}(2,\mathbb{Z})$ transformation we have

$$Q' = r_1r_2r_3(e_2 - ke_1), \qquad P' = -e_1 + dr_2r_3(e_2 - ke_1)\,. \tag{9}$$

We now define

$$e'_1 = (e_2 - ke_1), \qquad e'_2 = -e_1 + (dr_2r_3 - 1)(e_2 - ke_1)\,. \tag{10}$$

Since the matrix relating $(e'_1, e'_2)$ to $(e_1, e_2)$ has unit determinant, $(e'_1, e'_2)$ is a primitive basis of the lattice $\Lambda_0$. In this basis $(Q', P')$ can be expressed as

$$Q' = r_1r_2r_3e'_1, \qquad P' = e'_1 + e'_2\,. \tag{11}$$

Comparing this with (2) we see that for the new charge vector $(Q', P')$ we have

$$r'_1 = r_1r_2r_3, \quad r'_2 = 1, \quad r'_3 = 1, \quad u'_1 = 1\,. \tag{12}$$

This proves the desired result.

Next we shall study the subgroup of S-duality transformations which takes a configuration with $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ to another configuration with $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$. The initial configuration has

$$Q = re_1, \qquad P = e_1 + e_2\,. \tag{13}$$

An S-duality transformation (3) takes this to

$$Q' = are_1 + b(e_1 + e_2)\,, \qquad P' = cre_1 + d(e_1 + e_2)\,. \tag{14}$$

In order that $Q'$ is $r$ times a primitive vector, we must demand

$$b = 0 \bmod r\,. \tag{15}$$

Expressing $b$ as $b_0r$ with $b_0 \in \mathbb{Z}$ we get

$$Q' = re'_1, \qquad P' = e'_1 + e'_2\,, \tag{16}$$

where

$$e'_1 = (a + b_0)e_1 + b_0e_2, \qquad e'_2 = (cr + d - a - b_0)e_1 + (d - b_0)e_2\,. \tag{17}$$



JHEP04(2008)012

Since the determinant of the matrix relating $(e'_1, e'_2)$ to $(e_1, e_2)$ is given by

$$(a + b_0)(d - b_0) - b_0(cr + d - a - b_0) = ad - bc = 1\,, \tag{18}$$

we conclude that $(e'_1, e'_2)$ is a primitive basis of $\Lambda_0$. Comparison with (2) now shows that $(Q', P')$ has $r'_1 = r$, $r'_2 = r'_3 = u'_1 = 1$ as required. Thus the only condition on the $\mathrm{SL}(2,\mathbb{Z})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for preserving the $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ condition is that it must have $b = 0 \bmod r$, i.e. it must be an element of $\Gamma^0(r)$.

Using this we can now determine the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ that takes a pair of charge vectors $(Q, P)$ with invariants $(r_1, r_2, r_3, u_1)$ to another pair of charge vectors with the same invariants. For this we note that any $\mathrm{SL}(2,\mathbb{Z})$ transformation matrix $g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a$, $b$ given in (8) takes the set $(r_1, r_2, r_3, u_1)$ to the set $(r_1 r_2 r_3, 1, 1, 1)$. Since the latter set is preserved by the $\Gamma^0(r)$ subgroup of $\mathrm{SL}(2,\mathbb{Z})$, the original set must be preserved by the subgroup $g_0^{-1}\Gamma^0(r) g_0$. This is isomorphic to the group $\Gamma^0(r)$.

To see an example of this consider the case

$$r_1 = r_2 = 1, \quad r_3 = 2, \quad u_1 = 1\,. \tag{19}$$

In this case the $\mathrm{SL}(2,\mathbb{Z})$ transformation $g_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ takes a configuration given in (19) to a configuration with $r_1 = 2$, $r_2 = r_3 = u_1 = 1$. Thus the $\mathrm{SL}(2,\mathbb{Z})$ transformations which take a configuration with $(r_1 = 1, r_2 = 1, r_3 = 2, u_1 = 1)$ to a configuration with the same discrete invariants will be of the form:

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 2b_0 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a - c & a - c - d + 2b_0 \\ c & c + d \end{pmatrix}. \tag{20}$$

Since the condition $ad - 2b_0 c = 1$ requires $a$ and $d$ to be odd, we have

$$a' + b' \in 2\mathbb{Z} + 1, \qquad c' + d' \in 2\mathbb{Z} + 1\,. \tag{21}$$

Conversely given any $\mathrm{SL}(2,\mathbb{Z})$ matrix $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ satisfying (21), it can be written as $g_0$ conjugate of the $\Gamma^0(2)$ matrix $\begin{pmatrix} a' + c' & -a' - c' + b' + d' \\ c' & -c' + d' \end{pmatrix}$. Thus (21) characterizes the subgroup of S-duality group which preserves the condition (19).

The results derived so far make it clear that for a given torsion $r$ the discrete T-duality invariants are in one to one correspondence with the elements of the coset $\mathrm{SL}(2,\mathbb{Z})/\Gamma^0(r)$. The representative element for a given set of invariants $(r_1, r_2, r_3, u_1)$ is the element $g_0^{-1} \in \mathrm{SL}(2,\mathbb{Z})$ that takes a configuration with $(r_1 r_2 r_3, 1, 1, 1)$ to a configuration with discrete invariants $(r_1, r_2, r_3, u_1)$. Multiplying $g_0^{-1}$ by a $\Gamma^0(r)$ element from the right does not change the final values $(r_1, r_2, r_3, u_1)$ of the discrete invariants since a $\Gamma^0(r)$ transformation does not change the discrete T-duality invariants of the initial configuration.

We shall now examine the consequences of these results for the formula expressing the degeneracy $d(Q, P)$ — or more precisely an appropriate index measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets for a given set

JHEP04(2008)012

of charges[1] – of quarter BPS dyons as a function of $(Q, P)$. We note first of all that besides depending on $(Q, P)$, the degeneracy can also depend on the asymptotic values of the moduli fields, collectively denoted as $\phi$. We expect the dependence on $\phi$ to be mild, in the sense that the degeneracy formula should be $\phi$ independent within a given domain bounded by walls of marginal stability. It follows from the analysis of [7, 8] that the decays relevant for the walls of marginal stability are of the form

$$(Q, P) \to (\alpha Q + \beta P, \gamma Q + \delta P) + ((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P)\,, \qquad (22)$$

where $\alpha$, $\beta$, $\gamma$, $\delta$ are not necessarily integers, but must be such that $\alpha Q + \beta P$ and $\gamma Q + \delta P$ belong to the Narain lattice $\Lambda$. If we denote by $m(Q, P; \phi)$ the BPS mass of a dyon of charge $(Q, P)$ then the wall of marginal stability associated with the set $(\alpha, \beta, \gamma, \delta)$ is given by the solution to the equation

$$m(Q, P; \phi) = m(\alpha Q + \beta P, \gamma Q + \delta P; \phi) + m((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P; \phi)\,. \qquad (23)$$

For appropriate choice of $(\alpha, \beta, \gamma, \delta)$ this describes a codimension one subspace of the moduli space labelled by $\phi$. Since the BPS mass formula is invariant under a T-duality transformation $Q \to \Omega Q$, $P \to \Omega P$, $\phi \to \phi_\Omega$:

$$m(\Omega Q, \Omega P; \phi_\Omega) = m(Q, P; \phi) \qquad \Omega \in O(6, 22; \mathbb{Z})\,, \qquad (24)$$

eq.(23) may be written as

$$\begin{aligned} m(\Omega Q, \Omega P; \phi_\Omega) &= m(\alpha \Omega Q + \beta \Omega P, \gamma \Omega Q + \delta \Omega P; \phi_\Omega) \\ &\quad + m((1-\alpha)\Omega Q - \beta \Omega P, -\gamma \Omega Q + (1-\delta)\Omega P; \phi_\Omega)\,. \end{aligned} \qquad (25)$$

This is identical to eq.(23) with $(Q, P, \phi)$ replaced by $(\Omega Q, \Omega P, \phi_\Omega)$. This shows that under a T-duality transformation on charges and moduli, the wall of marginal stability associated with the set $(\alpha, \beta, \gamma, \delta)$ gets mapped to the wall of marginal stability associated with the same $(\alpha, \beta, \gamma, \delta)$. Thus if we consider a domain bounded by the walls of marginal stability associated with the sets $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ for $1 \le i \le n$ — collectively denoted by a set of discrete variables $\vec{c}$ — then under a simultaneous T-duality transformation on the charges and the moduli this domain gets mapped to a domain labelled by the same vector $\vec{c}$. The precise shape of the domain of course changes since the locations of the walls in the moduli space depends not only on $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ for $1 \le i \le n$ but also on the charges $(Q, P)$ which transform to $(\Omega Q, \Omega P)$.

We now use the fact that the dyon degeneracy formula must be invariant under a simultaneous T-duality transformation on the charges and the moduli, and also the fact that the dependence of $d(Q, P; \phi)$ on the moduli $\phi$ comes only through the domain in which $\phi$ lies, i.e. the vector $\vec{c}$. Since $\vec{c}$ remains unchanged under a T-duality transformation, we have

$$d(Q, P; \vec{c}) = d(\Omega Q, \Omega P; \vec{c})\,, \qquad \Omega \in O(6, 22; \mathbb{Z})\,. \qquad (26)$$

[1] Up to a normalization this is equal to the helicity trace $B_6 = Tr(-1)^{2h} h^6$ over all states carrying charge quantum numbers $(Q, P)$. Here $h$ denotes the helicity of the state.



JHEP04(2008)012

This shows that $d(Q,P;\vec{c})$ must depend only on $(Q,P)$ via the T-duality invariants:

$$d(Q,P;\vec{c}) = f(Q^2, P^2, Q\cdot P, r_1, r_2, r_3, u_1; \vec{c})\,, \tag{27}$$

for some function $f$.

Let us now study the effect of S-duality transformation on this formula. Typically an S-duality transformation will act on the charges and hence on all the T-duality invariants and also on the vector $\vec{c}$ labelling the domain bounded by the walls of marginal stability [9, 5, 10]. Indeed, as is clear from the condition (23), under an S-duality transformation of the form (3), the wall associated with the parameters $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ gets mapped to the wall associated with

$$\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}\,. \tag{28}$$

Thus S-duality invariance of the degeneracy formula now gives

$$f(Q^2, P^2, Q\cdot P, r_1, r_2, r_3, u_1; \vec{c}) = f(Q'^2, P'^2, Q'\cdot P', r_1', r_2', r_3', u_1'; \vec{c}\,')\,, \tag{29}$$

where $\vec{c}\,'$ stands for the collection of the sets $\{\alpha_i', \beta_i', \gamma_i', \delta_i'\}$ computed according to (28). We now use the result that there exists a special class of S-duality transformations under which

$$(r_1', r_2', r_3', u_1') = (r_1 r_2 r_3, 1, 1, 1)\,. \tag{30}$$

Using this S-duality transformation we get

$$f(Q^2, P^2, Q\cdot P, r_1, r_2, r_3, u_1; \vec{c}) = f(Q'^2, P'^2, Q'\cdot P', r_1 r_2 r_3, 1, 1, 1; \vec{c}\,')\,. \tag{31}$$

Thus the complete information about the spectrum of quarter BPS dyons is contained in the set of functions

$$g(Q^2, P^2, Q\cdot P, r; \vec{c}) \equiv f(Q^2, P^2, Q\cdot P, r, 1, 1, 1; \vec{c})\,. \tag{32}$$

We shall focus our attention on this function during the rest of our analysis. Using the fact that $\Gamma^0(r)$ transformations leave the set $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ fixed, we see that

$$\begin{aligned} & g(Q^2, P^2, Q\cdot P, r; \vec{c}) = g(Q'^2, P'^2, Q'\cdot P', r; \vec{c}\,') \\ \text{for} \quad & \begin{pmatrix}Q'\\ P'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}Q\\ P\end{pmatrix}, \quad \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma^0(r)\,. \end{aligned} \tag{33}$$

In other words, the function $g(Q^2, P^2, Q\cdot P, r; \vec{c})$ is expected to have manifest invariance under the $\Gamma^0(r)$ subgroup of S-duality transformations.

So far our discussion has been independent of any specific formula for the function $g(Q^2, P^2, Q\cdot P, r; \vec{c})$. For $r = 1$ dyons an explicit formula for the function $g$ has been found in a wide class of $\mathcal{N} = 4$ supersymmetric theories [5, 9−25]. In all the known examples the function $g$ is obtained as a contour integral of the inverse of an appropriate modular form of a subgroup of $\mathrm{Sp}(2,\mathbb{Z})$. In particular for heterotic string theory on $T^6$ the modular form is the well known Igusa cusp form of weight 10 of the full $\mathrm{Sp}(2,\mathbb{Z})$ group, with the S-duality



JHEP04(2008)012

group $\mathrm{SL}(2,\mathbb{Z})$ embedded in $\mathrm{Sp}(2,\mathbb{Z})$ in a specific manner. Furthermore the dependence on the domain labelled by $\vec{c}$ is encoded fully in the choice of the integration contour and not in the integrand. If a similar formula exists for $g(Q^2, P^2, Q \cdot P, r; \vec{c})$ for $r > 1$, then our analysis would suggest that the integrand should involve a modular form of a subgroup of $\mathrm{Sp}(2,\mathbb{Z})$ that contains $\Gamma^0(r)$ in the same way that the full $\mathrm{Sp}(2,\mathbb{Z})$ contains $\mathrm{SL}(2,\mathbb{Z})$. It remains to be seen if this constraint together with other physical constraints reviewed in [25] can fix the form of the integrand.

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JHEP04(2008)012

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JHEP04(2008)012